Functions and Limits

Numbers and Functions

The subject of this course is "functions of one real variable" so we begin by wondering what a real number "really" is, and then, in the next section, what a function is.

1. What is a number?

1.1. Different kinds of numbers. The simplest numbers are the ***positive integers***

$$1, 2, 3, 4, \cdots$$

the number ***zero***

$$0,$$

and the ***negative integers***

$$\cdots, -4, -3, -2, -1.$$

Together these form the integers or "whole numbers."

Next, there are the numbers you get by dividing one whole number by another (nonzero) whole number. These are the so called fractions or ***rational numbers*** such as

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{3}, \cdots$$

or

$$-\frac{1}{2}, -\frac{1}{3}, -\frac{2}{3}, -\frac{1}{4}, -\frac{2}{4}, -\frac{3}{4}, -\frac{4}{3}, \cdots$$

By definition, any whole number is a rational number (in particular zero is a rational number.)

You can add, subtract, multiply and divide any pair of rational numbers and the result will again be a rational number (provided you don't try to divide by zero).

One day in middle school you were told that there are other numbers besides the rational numbers, and the first example of such a number is the square root of two. It has been known ever since the time of the greeks that no rational number exists whose square is exactly 2, i.e. you can't find a fraction $\frac{m}{n}$ such that

$$\left(\frac{m}{n}\right)^2 = 2, \text{ i.e. } m^2 = 2n^2.$$

x	x^2
1.2	1.44
1.3	1.69
1.4	$1.96 < 2$
1.5	$2.25 > 2$
1.6	2.56

Nevertheless, if you compute x^2 for some values of x between 1 and 2, and check if you get more or less than 2, then it looks like there should be some number x between 1.4 and 1.5 whose square is exactly 2. So, we *assume* that there is such a number, and we call it the square root of 2, written as $\sqrt{2}$. This raises several questions. How do we know there really is a number between 1.4 and 1.5 for which $x^2 = 2$? How many other such numbers are we going to assume into existence? Do these new numbers obey the same algebra rules (like $a + b = b + a$) as the rational numbers? If we knew precisely what these numbers (like $\sqrt{2}$) were then we could perhaps answer such questions. It turns out to be rather difficult to give a precise description of what a number is, and in this course we won't try to get anywhere near the bottom of this issue. Instead we will think of numbers as "infinite decimal expansions" as follows.

One can represent certain fractions as decimal fractions, e.g.

$$\frac{279}{25} = \frac{1116}{100} = 11.16.$$

Not all fractions can be represented as decimal fractions. For instance, expanding $\frac{1}{3}$ into a decimal fraction leads to an unending decimal fraction

$$\frac{1}{3} = 0.333\,333\,333\,333\,333\,\cdots$$

It is impossible to write the complete decimal expansion of $\frac{1}{3}$ because it contains infinitely many digits. But we can describe the expansion: each digit is a three. An electronic calculator, which always represents numbers as ***finite*** decimal numbers, can never hold the number $\frac{1}{3}$ exactly.

Every fraction can be written as a decimal fraction which may or may not be finite. If the decimal expansion doesn't end, then it must repeat. For instance,

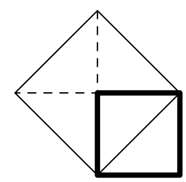
$$\frac{1}{7} = 0.142857\,142857\,142857\,142857\,\ldots$$

Conversely, any infinite repeating decimal expansion represents a rational number.

A ***real number*** is specified by a possibly unending decimal expansion. For instance,

$$\sqrt{2} = 1.414\,213\,562\,373\,095\,048\,801\,688\,724\,209\,698\,078\,569\,671\,875\,376\,9\ldots$$

Of course you can never write *all* the digits in the decimal expansion, so you only write the first few digits and hide the others behind dots. To give a precise description of a real number (such as $\sqrt{2}$) you have to explain how you could *in principle* compute as many digits in the expansion as you would like. During the next three semesters of calculus we will not go into the details of how this should be done.



1.2. A reason to believe in $\sqrt{2}$. The Pythagorean theorem says that the hypotenuse of a right triangle with sides 1 and 1 must be a line segment of length $\sqrt{2}$. In middle or high school you learned something similar to the following geometric construction of a line segment whose length is $\sqrt{2}$. Take a square with side of length 1, and construct a new square one of whose sides is the diagonal of the first square. The figure you get consists of 5 triangles of equal area and by counting triangles you see that the larger square has exactly twice the area of the smaller square. Therefore the diagonal of the smaller square, being the side of the larger square, is $\sqrt{2}$ as long as the side of the smaller square.

Why are real numbers called real? All the numbers we will use in this first semester of calculus are "real numbers." At some point (in 2nd semester calculus) it becomes useful to assume that there is a number whose square is -1. No real number has this property since the square of any real number is positive, so it was decided to call this new imagined number "imaginary" and to refer to the numbers we already have (rationals, $\sqrt{2}$-like things) as "real."

1.3. The real number line and intervals. It is customary to visualize the real numbers as points on a straight line. We imagine a line, and choose one point on this line, which we call the ***origin***. We also decide which direction we call "left" and hence which we call "right." Some draw the number line vertically and use the words "up" and "down."

To plot any real number x one marks off a distance x from the origin, to the right (up) if $x > 0$, to the left (down) if $x < 0$.

The ***distance along the number line*** between two numbers x and y is $|x - y|$. In particular, the distance is never a negative number.

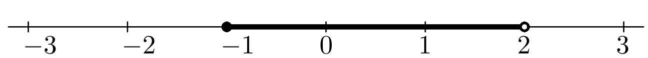


Figure 1. To draw the half open interval $[-1, 2)$ use a filled dot to mark the endpoint which is included and an open dot for an excluded endpoint.

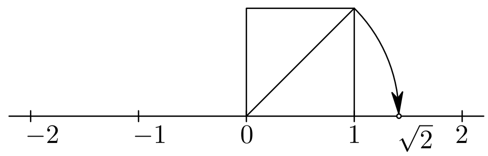


Figure 2. To find $\sqrt{2}$ on the real line you draw a square of sides 1 and drop the diagonal onto the real line.

Almost every equation involving variables x, y, etc. we write down in this course will be true for some values of x but not for others. In modern abstract mathematics a collection of real numbers (or any other kind of mathematical objects) is called a ***set***. Below are some examples of sets of real numbers. We will use the notation from these examples throughout this course.

The collection of all real numbers between two given real numbers form an interval. The following notation is used

- (a, b) is the set of all real numbers x which satisfy $a < x < b$.
- $[a, b)$ is the set of all real numbers x which satisfy $a \le x < b$.
- $(a, b]$ is the set of all real numbers x which satisfy $a < x \le b$.
- $[a, b]$ is the set of all real numbers x which satisfy $a \le x \le b$.

If the endpoint is not included then it may be ∞ or $-\infty$. E.g. $(-\infty, 2]$ is the interval of all real numbers (both positive and negative) which are ≤ 2.

1.4. Set notation. A common way of describing a set is to say it is the collection of all real numbers which satisfy a certain condition. One uses this notation

$$\mathcal{A} = \{x \mid x \text{ satisfies this or that condition}\}$$

Most of the time we will use upper case letters in a calligraphic font to denote sets. ($\mathcal{A}$,$\mathcal{B}$,$\mathcal{C}$,$\mathcal{D}$, ...)

For instance, the interval (a, b) can be described as

$$(a, b) = \{x \mid a < x < b\}$$

The set

$$\mathcal{B} = \{x \mid x^2 - 1 > 0\}$$

consists of all real numbers x for which $x^2 - 1 > 0$, i.e. it consists of all real numbers x for which either $x > 1$ or $x < -1$ holds. This set consists of two parts: the interval $(-\infty, -1)$ and the interval $(1, \infty)$.

You can try to draw a set of real numbers by drawing the number line and coloring the points belonging to that set red, or by marking them in some other way.

Some sets can be very difficult to draw. For instance,

$$\mathcal{C} = \{x \mid x \text{ is a rational number}\}$$

can't be accurately drawn. In this course we will try to avoid such sets.

Sets can also contain just a few numbers, like

$$\mathcal{D} = \{1, 2, 3\}$$

which is the set containing the numbers one, two and three. Or the set

$$\mathcal{E} = \{x \mid x^3 - 4x^2 + 1 = 0\}$$

which consists of the solutions of the equation $x^3 - 4x^2 + 1 = 0$. (There are three of them, but it is not easy to give a formula for the solutions.)

If $\mathcal{A}$ and $\mathcal{B}$ are two sets then ***the union of*** $\mathcal{A}$ ***and*** $\mathcal{B}$ is the set which contains all numbers that belong either to $\mathcal{A}$ or to $\mathcal{B}$. The following notation is used

$$\mathcal{A} \cup \mathcal{B} = \{x \mid x \text{ belongs to } \mathcal{A} \text{ or to } \mathcal{B} \text{ or both.}\}$$

Similarly, the ***intersection of two sets $\mathcal{A}$ and $\mathcal{B}$*** is the set of numbers which belong to both sets. This notation is used:

$$\mathcal{A} \cap \mathcal{B} = \{x \mid x \text{ belongs to both } \mathcal{A} \text{ and } \mathcal{B}.\}$$

2. Exercises

1. What is the 2007^{th} digit after the period in the expansion of $\frac{1}{7}$?

2. Which of the following fractions have finite decimal expansions?

$$a = \frac{2}{3}, \quad b = \frac{3}{25}, \quad c = \frac{276937}{15625}.$$

3. Draw the following sets of real numbers. Each of these sets is the union of one or more intervals. Find those intervals. Which of thee sets are finite?

$\mathcal{A} = \{x \mid x^2 - 3x + 2 \le 0\}$
$\mathcal{B} = \{x \mid x^2 - 3x + 2 \ge 0\}$
$\mathcal{C} = \{x \mid x^2 - 3x > 3\}$
$\mathcal{D} = \{x \mid x^2 - 5 > 2x\}$
$\mathcal{E} = \{t \mid t^2 - 3t + 2 \le 0\}$
$\mathcal{F} = \{\alpha \mid \alpha^2 - 3\alpha + 2 \ge 0\}$
$\mathcal{G} = (0,1) \cup (5,7]$
$\mathcal{H} = (\{1\} \cup \{2,3\}) \cap (0, 2\sqrt{2})$
$\mathcal{Q} = \{\theta \mid \sin\theta = \frac{1}{2}\}$
$\mathcal{R} = \{\varphi \mid \cos\varphi > 0\}$

4. Suppose $\mathcal{A}$ and $\mathcal{B}$ are intervals. Is it always true that $\mathcal{A} \cap \mathcal{B}$ is an interval? How about $\mathcal{A} \cup \mathcal{B}$?

5. Consider the sets

$$\mathcal{M} = \{x \mid x > 0\} \text{ and } \mathcal{N} = \{y \mid y > 0\}.$$

Are these sets the same?

6. Group Problem.

Write the numbers

$$x = 0.3131313131\ldots, \quad y = 0.273273273273\ldots$$
$$\text{and } z = 0.21541541541541541\ldots$$

as fractions (i.e. write them as $\frac{m}{n}$, specifying m and n.)

(Hint: show that $100x = x + 31$. A similar trick works for y, but z is a little harder.)

7. Group Problem.

Is the number whose decimal expansion after the period consists only of nines, i.e.

$$x = 0.99999999999999999\ldots$$

an integer?

3. Functions

Wherein we meet the main characters of this semester

3.1. Definition. To specify a ***function*** f you must

(1) give a ***rule*** which tells you how to compute the value $f(x)$ of the function for a given real number x, and:
(2) say for which real numbers x the rule may be applied.

The set of numbers for which a function is defined is called its ***domain***. The set of all possible numbers $f(x)$ as x runs over the domain is called the ***range*** of the function. The rule must be ***unambiguous:*** the same $x$must always lead to the same $f(x)$.

For instance, one can define a function f by putting $f(x) = \sqrt{x}$ for all $x \ge 0$. Here the rule defining f is "take the square root of whatever number you're given", and the function f will accept all nonnegative real numbers.

The rule which specifies a function can come in many different forms. Most often it is a formula, as in the square root example of the previous paragraph. Sometimes you need a few formulas, as in

$$g(x) = \begin{cases} 2x & \text{for } x < 0 \\ x^2 & \text{for } x \ge 0 \end{cases} \qquad \text{domain of } g = \text{all real numbers.}$$

Functions which are defined by different formulas on different intervals are sometimes called ***piecewise defined functions.***

3.2. Graphing a function. You get the ***graph of a function*** f by drawing all points whose coordinates are (x, y) where x must be in the domain of f and $y = f(x)$.

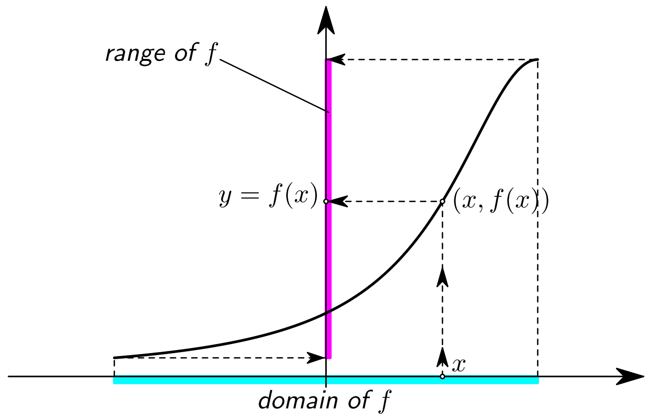


Figure 3. The graph of a function f. The domain of f consists of all x values at which the function is defined, and the range consists of all possible values f can have.

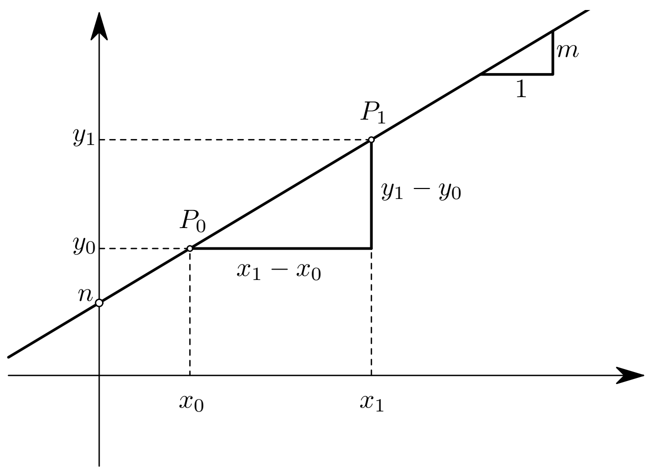


Figure 4. A straight line and its slope. The line is the graph of $f(x) = mx + n$. It intersects the y-axis at height n, and the ratio between the amounts by which y and x increase as you move from one point to another on the line is $\frac{y_1 - y_0}{x_1 - x_0} = m$.

3.3. Linear functions. A function which is given by the formula

$$f(x) = mx + n$$

where m and n are constants is called a ***linear function***. Its graph is a straight line. The constants m and n are the ***slope*** and ***y-intercept*** of the line. Conversely, any straight line which is not vertical (i.e. not parallel to the y-axis) is the graph of a linear function. If you know two points (x_0, y_0) and (x_1, y_1) on the line, then then one can compute the slope m from the "rise-over-run" formula

$$m = \frac{y_1 - y_0}{x_1 - x_0}.$$

This formula actually contains a theorem from Euclidean geometry, namely it says that the ratio $(y_1 - y_0) : (x_1 - x_0)$ is the same for every pair of points (x_0, y_0) and (x_1, y_1) that you could pick on the line.

3.4. Domain and "biggest possible domain. " In this course we will usually not be careful about specifying the domain of the function. When this happens the domain is understood to be the set of all x for which the rule which tells you how to compute $f(x)$ is meaningful. For instance, if we say that h is the function

$$h(x) = \sqrt{x}$$

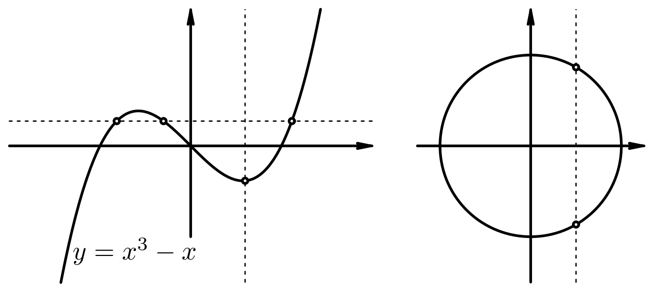


Figure 5. The graph of $y = x^3 - x$ fails the "horizontal line test," but it passes the "vertical line test." The circle fails both tests.

then the domain of h is understood to be the set of all nonnegative real numbers

$$\text{domain of } h = [0, \infty)$$

since $\sqrt{x}$ is well-defined for all $x \geq 0$ and undefined for $x < 0$.

A systematic way of finding the domain and range of a function for which you are only given a formula is as follows:

- The domain of f consists of all x for which $f(x)$ is well-defined ("makes sense")
- The range of f consists of all y for which you can solve the equation $f(x) = y$.

3.5. Example – find the domain and range of $f(x) = 1/x^2$. The expression $1/x^2$ can be computed for all real numbers x except $x = 0$ since this leads to division by zero. Hence the domain of the function $f(x) = 1/x^2$ is

$$\text{“all real numbers except 0”} = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty).$$

To find the range we ask "for which y can we solve the equation $y = f(x)$ for x," i.e. we for which y can you solve $y = 1/x^2$ for x?

If $y = 1/x^2$ then we must have $x^2 = 1/y$, so first of all, since we have to divide by y, y can't be zero. Furthermore, $1/y = x^2$ says that y must be positive. On the other hand, if $y > 0$ then $y = 1/x^2$ has a solution (in fact two solutions), namely $x = \pm 1/\sqrt{y}$. This shows that the range of f is

$$\text{“all positive real numbers”} = \{x \mid x > 0\} = (0, \infty).$$

3.6. Functions in "real life. " One can describe the motion of an object using a function. If some object is moving along a straight line, then you can define the following function: Let $x(t)$ be the distance from the object to a fixed marker on the line, at the time t. Here the domain of the function is the set of all times t for which we know the position of the object, and the rule is

Given t, measure the distance between the object and the marker at time t.

There are many examples of this kind. For instance, a biologist could describe the growth of a cell by defining $m(t)$ to be the mass of the cell at time t (measured since the birth of the cell). Here the domain is the interval $[0, T]$, where T is the life time of the cell, and the rule that describes the function is

Given t, weigh the cell at time t.

3.7. The Vertical Line Property. Generally speaking graphs of functions are curves in the plane but they distinguish themselves from arbitrary curves by the way they intersect vertical lines: ***The graph of a function cannot intersect a vertical line "$x =$ constant" in more than one point.*** The reason why this is true is very simple: if two points lie on a vertical line, then they have the same x coordinate, so if they also lie on the graph of a function f, then their y-coordinates must also be equal, namely $f(x)$.

3.8. Examples. The graph of $f(x) = x^3 - x$ "goes up and down," and, even though it intersects several horizontal lines in more than one point, it intersects ***every*** vertical line in exactly one point.

The collection of points determined by the equation $x^2 + y^2 = 1$ is a circle. It is not the graph of a function since the vertical line $x = 0$ (the y-axis) intersects the graph in two points $P_1(0, 1)$ and $P_2(0, -1)$. See Figure 6.

4. Inverse functions and Implicit functions

For many functions the rule which tells you how to compute it is not an explicit formula, but instead an equation which you still must solve. A function which is defined in this way is called an "implicit function."

4.1. Example. One can define a function f by saying that for each x the value of $f(x)$ is the solution y of the equation

$$x^2 + 2y - 3 = 0.$$

In this example you can solve the equation for y,

$$y = \frac{3 - x^2}{2}.$$

Thus we see that the function we have defined is $f(x) = (3 - x^2)/2$.

Here we have two definitions of the same function, namely

(i) "$y = f(x)$ is defined by $x^2 + 2y - 3 = 0$," and
(ii) "f is defined by $f(x) = (3 - x^2)/2$."

The first definition is the implicit definition, the second is explicit. You see that with an "implicit function" it isn't the function itself, but rather the way it was defined that's implicit.

4.2. Another example: domain of an implicitly defined function. Define g by saying that for any x the value $y = g(x)$ is the solution of

$$x^2 + xy - 3 = 0.$$

Just as in the previous example one can then solve for y, and one finds that

$$g(x) = y = \frac{3 - x^2}{x}.$$

Unlike the previous example this formula does not make sense when $x = 0$, and indeed, for $x = 0$ our rule for g says that $g(0) = y$ is the solution of

$$0^2 + 0 \cdot y - 3 = 0, \text{ i.e. } y \text{ is the solution of } 3 = 0.$$

That equation has no solution and hence $x = 0$ does not belong to the domain of our function g.

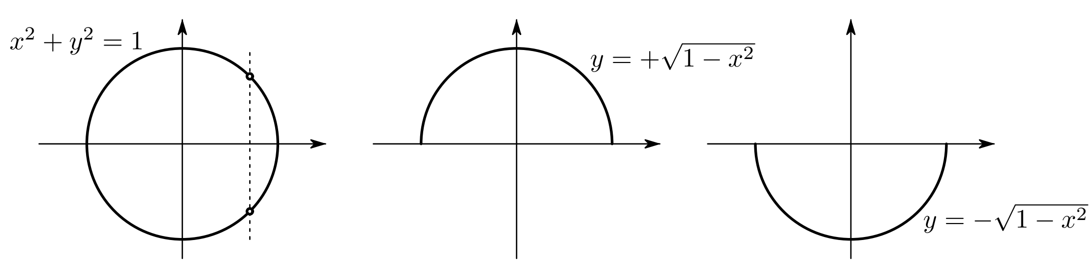


Figure 6. The circle determined by $x^2 + y^2 = 1$ is not the graph of a function, but it contains the graphs of the two functions $h_1(x) = \sqrt{1 - x^2}$ and $h_2(x) = -\sqrt{1 - x^2}$.

Limits and Continuous Functions

1. Informal definition of limits

While it is easy to define precisely in a few words what a square root is ($\sqrt{a}$ is the positive number whose square is a) the definition of the limit of a function runs over several terse lines, and most people don't find it very enlightening when they first see it. (See §2.) So we postpone this for a while and fine tune our intuition for another page.

1.1. Definition of limit (1st attempt). If f is some function then

$$\lim_{x\to a} f(x) = L$$

is read "the limit of $f(x)$ as x approaches a is L." It means that if you choose values of x which are close ***but not equal*** to a, then $f(x)$ will be close to the value L; moreover, $f(x)$ gets closer and closer to L as x gets closer and closer to a.

The following alternative notation is sometimes used

$$f(x) \to L \quad \text{as} \quad x \to a;$$

(read "$f(x)$ approaches L as x approaches a" or "$f(x)$ goes to L is x goes to a".)

1.2. Example. If $f(x) = x + 3$ then

$$\lim_{x\to 4} f(x) = 7,$$

is true, because if you substitute numbers x close to 4 in $f(x) = x + 3$ the result will be close to 7.

1.3. Example: substituting numbers to guess a limit. What (if anything) is

$$\lim_{x\to 2} \frac{x^2 - 2x}{x^2 - 4}?$$

Here $f(x) = (x^2 - 2x)/(x^2 - 4)$ and $a = 2$.

We first try to substitute $x = 2$, but this leads to

$$f(2) = \frac{2^2 - 2\cdot 2}{2^2 - 4} = \frac{0}{0}$$

which does not exist. Next we try to substitute values of x close but not equal to 2. Table 1 suggests that $f(x)$ approaches 0.5.

x	$f(x)$	x	$g(x)$
3.000000	0.600000	1.000000	1.009990
2.500000	0.555556	0.500000	1.009980
2.100000	0.512195	0.100000	1.009899
2.010000	0.501247	0.010000	1.008991
2.001000	0.500125	0.001000	1.000000

Table 1. Finding limits by substituting values of x "close to a." (Values of $f(x)$ and $g(x)$ rounded to six decimals.)

1.4. Example: Substituting numbers can suggest the wrong answer. The previous example shows that our first definition of "limit" is not very precise, because it says "x close to a," but how close is close enough? Suppose we had taken the function

$$g(x) = \frac{101\,000x}{100\,000x + 1}$$

and we had asked for the limit $\lim_{x\to 0} g(x)$.

Then substitution of some "small values of x" could lead us to believe that the limit is $1.000\ldots$. Only when you substitute even smaller values do you find that the limit is 0 (zero)!

See also problem 29.

2. The formal, authoritative, definition of limit

The informal description of the limit uses phrases like "closer and closer" and "really very small." In the end we don't really know what they mean, although they are suggestive. "Fortunately" there is a good definition, i.e. one which is unambiguous and can be used to settle any dispute about the question of whether $\lim_{x\to a} f(x)$ equals some number L or not. Here is the definition. It takes a while to digest, so read it once, look at the examples, do a few exercises, read the definition again. Go on to the next sections. Throughout the semester come back to this section and read it again.

2.1. Definition of $\lim_{x\to a} f(x) = L$. *We say that L is the limit of $f(x)$ as $x \to a$, if*

(1) *$f(x)$ need not be defined at $x = a$, but it must be defined for all other x in some interval which contains a.*
(2) *for every $\varepsilon > 0$ one can find a $\delta > 0$ such that for all x in the domain of f one has*

$$|x - a| < \delta \text{ implies } |f(x) - L| < \varepsilon. \tag{8}$$

Why the absolute values? The quantity $|x - y|$ is the distance between the points x and y on the number line, and one can measure how close x is to y by calculating $|x - y|$. The inequality $|x - y| < \delta$ says that "the distance between x and y is less than δ," or that "x and y are closer than δ."

What are ε and δ? The quantity ε is how close you would like $f(x)$ to be to its limit L; the quantity δ is how close you have to choose x to a to achieve this. To prove that $\lim_{x\to a} f(x) = L$ you must assume that someone has given you an unknown $\varepsilon > 0$, and then find a postive δ for which (8) holds. The δ you find will depend on ε.

2.2. Show that $\lim_{x\to 5} 2x + 1 = 11$. We have $f(x) = 2x + 1$, $a = 5$ and $L = 11$, and the question we must answer is "how close should x be to 5 if want to be sure that $f(x) = 2x + 1$ differs less than ε from $L = 11$?"

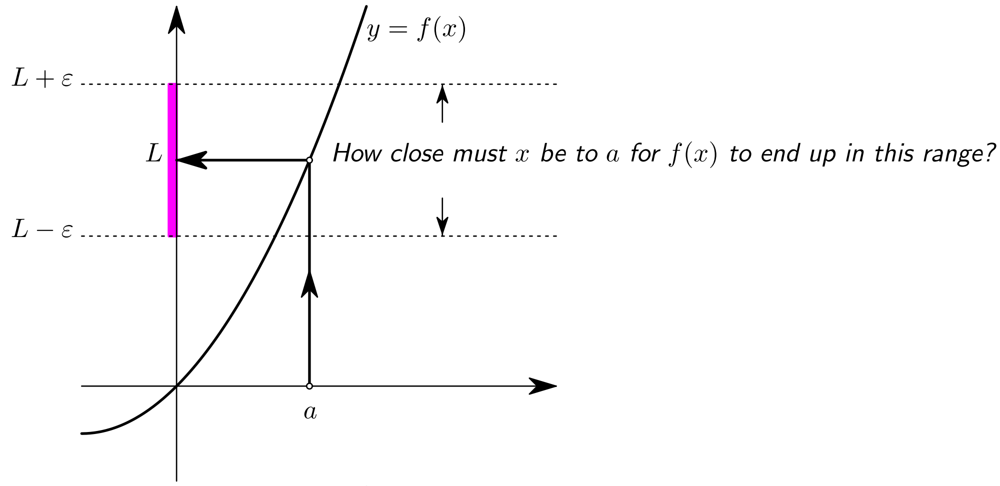
To figure this out we try to get an idea of how big $|f(x) - L|$ is:

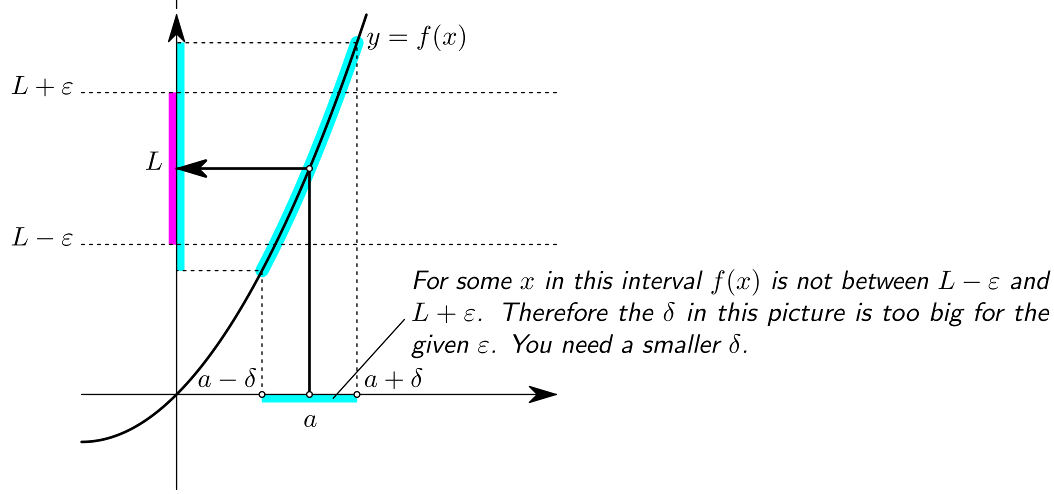
$$|f(x) - L| = \big|(2x + 1) - 11\big| = |2x - 10| = 2 \cdot |x - 5| = 2 \cdot |x - a|.$$

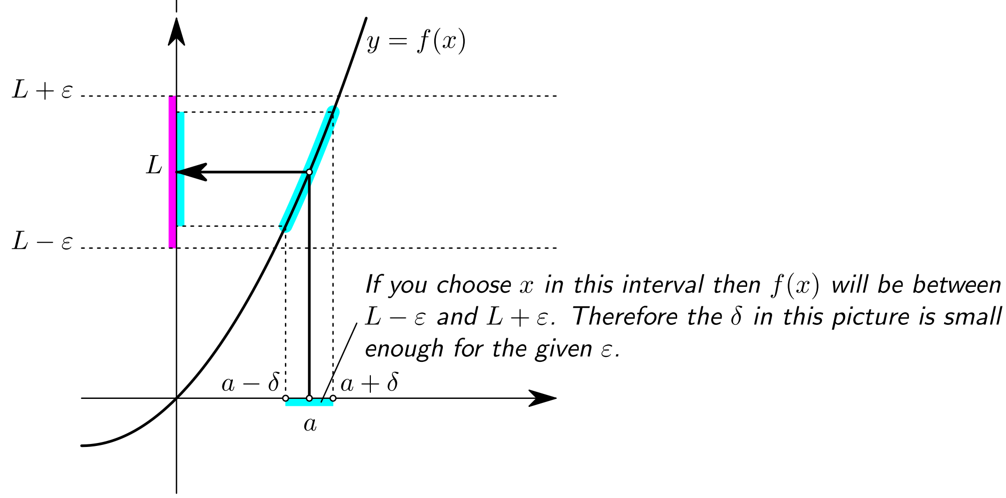
So, if $2|x - a| < \varepsilon$ then we have $|f(x) - L| < \varepsilon$, i.e.

$$\text{if } |x - a| < \tfrac{1}{2}\varepsilon \text{ then } |f(x) - L| < \varepsilon.$$

We can therefore choose $\delta = \frac{1}{2}\varepsilon$. No matter what $\varepsilon > 0$ we are given our δ will also be positive, and if $|x - 5| < \delta$ then we can guarantee $|(2x + 1) - 11| < \varepsilon$. That shows that $\lim_{x\to 5} 2x + 1 = 11$.







2.3. The limit $\lim_{x\to 1} x^2 = 1$ and the "don't choose $\delta > 1$" trick. We have $f(x) = x^2$, $a = 1$, $L = 1$, and again the question is, "how small should $|x - 1|$ be to guarantee $|x^2 - 1| < \varepsilon$?"

We begin by estimating the difference $|x^2 - 1|$

$$|x^2 - 1| = |(x - 1)(x + 1)| = |x + 1| \cdot |x - 1|.$$

Propagation of errors – another interpretation of ε and δ

According to the limit definition "$\lim_{x\to R} \pi x^2 = A$" is true if *for every $\varepsilon > 0$ you can find a $\delta > 0$ such that $|x - R| < \delta$ implies $|\pi x^2 - A| < \varepsilon$.* Here's a more concrete situation in which ε and δ appear in exactly the same roles:

Suppose you are given a circle drawn on a piece of paper, and you want to know its area. You decide to measure its radius, R, and then compute the area of the circle by calculating

$$\text{Area} = \pi R^2.$$

The area is a function of the radius, and we'll call that function f:

$$f(x) = \pi x^2.$$

When you measure the radius R you will make an error, simply because you can never measure anything with infinite precision. Suppose that R is the real value of the radius, and that x is the number you measured. Then the size of the error you made is

$$\text{error in radius measurement} = |x - R|.$$

When you compute the area you also won't get the exact value: you would get $f(x) = \pi x^2$ instead of $A = f(R) = \pi R^2$. The error in your computed value of the area is

$$\text{error in area } = |f(x) - f(R)| = |f(x) - A|.$$

Now you can ask the following question:

Suppose you want to know the area
with an error of at most ε,
then what is the largest error
that you can afford to make
when you measure the radius?

The answer will be something like this: if you want the computed area to have an error of at most $|f(x) - A| < \varepsilon$, then the error in your radius measurement should satisfy $|x - R| < \delta$. You have to do the algebra with inequalities to compute δ when you know ε, as in the examples in this section.

You would expect that if your measured radius x is close enough to the real value R, then your computed area $f(x) = \pi x^2$ will be close to the real area A.

In terms of ε and δ this means that you would expect that no matter how accurately you want to know the area (i.e how small you make ε) you can always achieve that precision by making the error in your radius measurement small enough (i.e. by making δ sufficiently small).

As x approaches 1 the factor $|x - 1|$ becomes small, and if the other factor $|x + 1|$ were a constant (e.g. 2 as in the previous example) then we could find δ as before, by dividing ε by that constant.

Here is a trick that allows you to replace the factor $|x + 1|$ with a constant. We hereby agree *that we always choose our δ so that $\delta \le 1$.* If we do that, then we will always have

$$|x - 1| < \delta \le 1, \text{i.e. } |x - 1| < 1,$$

and x will always be beween 0 and 2. Therefore

$$|x^2 - 1| = |x + 1| \cdot |x - 1| < 3|x - 1|.$$

If we now want to be sure that $|x^2 - 1| < \varepsilon$, then this calculation shows that we should require $3|x - 1| < \varepsilon$, i.e. $|x - 1| < \frac{1}{3}\varepsilon$. So we should choose $\delta \le \frac{1}{3}\varepsilon$. We must also live up to our promise never to choose $\delta > 1$, so if we are handed an ε for which $\frac{1}{3}\varepsilon > 1$, then we choose $\delta = 1$ instead of $\delta = \frac{1}{3}\varepsilon$. To summarize, we are going to choose

$$\delta = \text{the smaller of 1 and } \frac{1}{3}\varepsilon.$$

We have shown that if you choose δ this way, then $|x - 1| < \delta$ implies $|x^2 - 1| < \varepsilon$, no matter what $\varepsilon > 0$ is.

The expression "the smaller of a and b" shows up often, and is abbreviated to $\min(a, b)$. We could therefore say that in this problem we will choose δ to be

$$\delta = \min\bigl(1, \tfrac{1}{3}\varepsilon\bigr).$$

2.4. Show that $\lim_{x\to 4} 1/x = 1/4$**.** Solution: We apply the definition with $a = 4$, $L = 1/4$ and $f(x) = 1/x$. Thus, for any $\varepsilon > 0$ we try to show that if $|x - 4|$ is small enough then one has $|f(x) - 1/4| < \varepsilon$.

We begin by estimating $|f(x) - \frac{1}{4}|$ in terms of $|x - 4|$:

$$|f(x) - 1/4| = \left|\frac{1}{x} - \frac{1}{4}\right| = \left|\frac{4-x}{4x}\right| = \frac{|x-4|}{|4x|} = \frac{1}{|4x|}\,|x-4|.$$

As before, things would be easier if $1/|4x|$ were a constant. To achieve that we again agree not to take $\delta > 1$. If we always have $\delta \le 1$, then we will always have $|x - 4| < 1$, and hence $3 < x < 5$. How large can $1/|4x|$ be in this situation? Answer: the quantity $1/|4x|$ increases as you decrease x, so if $3 < x < 5$ then it will never be larger than $1/|4 \cdot 3| = \frac{1}{12}$.

We see that if we never choose $\delta > 1$, we will always have

$$|f(x) - \tfrac{1}{4}| \le \tfrac{1}{12}|x - 4| \quad \text{for} \quad |x - 4| < \delta.$$

To guarantee that $|f(x) - \frac{1}{4}| < \varepsilon$ we could threfore require

$$\tfrac{1}{12}|x - 4| < \varepsilon, \quad \text{i.e.} \quad |x - 4| < 12\varepsilon.$$

Hence if we choose $\delta = 12\varepsilon$ or any smaller number, then $|x - 4| < \delta$ implies $|f(x) - 4| < \varepsilon$. Of course we have to honor our agreement never to choose $\delta > 1$, so our choice of δ is

$$\delta = \text{the smaller of 1 and } 12\varepsilon = \min\bigl(1, 12\varepsilon\bigr).$$

3. Exercises

38. Group Problem.

Joe offers to make square sheets of paper for Bruce. Given $x > 0$ Joe plans to mark off a length x and cut out a square of side x. Bruce asks Joe for a square with area 4 square foot. Joe tells Bruce that he can't measure **exactly** 2 foot and the area of the square he produces will only be approximately 4 square foot. Bruce doesn't mind as long as the area of the square doesn't differ more than 0.01 square foot from what he really asked for (namely, 4 square foot).

(a) What is the biggest error Joe can afford to make when he marks off the length x?

(b) Jen also wants square sheets, with area 4 square feet. However, she needs the error in the area to be less than 0.00001 square foot. (She's paying).

How accurate must Joe measure the side of the squares he's going to cut for Jen?

Use the ε–δ definition to prove the following limits

39. $\lim\limits_{x\to 1} 2x - 4 = 6$

40. $\lim\limits_{x\to 2} x^2 = 4.$

41. $\lim\limits_{x\to 2} x^2 - 7x + 3 = -7$

42. $\lim\limits_{x\to 3} x^3 = 27$

43. $\lim\limits_{x\to 2} x^3 + 6x^2 = 32.$

44. $\lim\limits_{x\to 4} \sqrt{x} = 2.$

45. $\lim\limits_{x\to 3} \sqrt{x+6} = 9.$

46. $\lim\limits_{x\to 2} \dfrac{1+x}{4+x} = \frac{1}{2}.$

47. $\lim\limits_{x\to 1} \dfrac{2-x}{4-x} = \frac{1}{3}.$

48. $\lim\limits_{x\to 3} \dfrac{x}{6-x} = 1.$

49. $\lim\limits_{x\to 0} \sqrt{|x|} = 0$

50. Group Problem.

(Joe goes cubic.) Joe is offering to build cubes of side x. Airline regulations allow you take a cube on board provided its volume and surface area add up to less than 33 (everything measured in feet). For instance, a cube with 2 foot sides has volume+area equal to $2^3 + 6 \times 2^2 = 32$.

If you ask Joe to build a cube whose volume plus total surface area is 32 cubic feet with an error of at most ε, then what error can he afford to make when he measures the side of the cube he's making?

51. Our definition of a derivative in (7) contains a limit. What is the function "f" there, and what is the variable?

4. Variations on the limit theme

Not all limits are "for $x \to a$." here we describe some possible variations on the concept of limit.

4.1. Left and right limits. When we let "x approach a" we allow x to be both larger or smaller than a, as long as x gets close to a. If we explicitly want to study the behaviour of $f(x)$ as x approaches a through values larger than a, then we write

$$\lim_{x\searrow a} f(x) \text{ or } \lim_{x\to a+} f(x) \text{ or } \lim_{x\to a+0} f(x) \text{ or } \lim_{x\to a, x>a} f(x).$$

All four notations are in use. Similarly, to designate the value which $f(x)$ approaches as x approaches a through values below a one writes

$$\lim_{x\nearrow a} f(x) \text{ or } \lim_{x\to a-} f(x) \text{ or } \lim_{x\to a-0} f(x) \text{ or } \lim_{x\to a, x<a} f(x).$$

The precise definition of right limits goes like this:

4.2. Definition of right-limits. *Let f be a function. Then*

$$\lim_{x\searrow a} f(x) = L. \tag{9}$$

means that for every $\varepsilon > 0$ one can find a $\delta > 0$ such that

$$a < x < a + \delta \implies |f(x) - L| < \varepsilon$$

holds for all x in the domain of f.

The left-limit, i.e. the one-sided limit in which x approaches a through values less than a is defined in a similar way. The following theorem tells you how to use one-sided limits to decide if a function $f(x)$ has a limit at $x = a$.

4.3. Theorem. *If both one-sided limits*

$$\lim_{x\searrow a} f(x) = L_+, \text{ and } \lim_{x\nearrow a} f(x) = L_-$$

exist, then

$$\lim_{x\to a} f(x) \text{ exists} \iff L_+ = L_-.$$

In other words, if a function has both left- and right-limits at some $x = a$, then that function has a limit at $x = a$ if the left- and right-limits are equal.

4.4. Limits at infinity. Instead of letting x approach some finite number, one can let x become "larger and larger" and ask what happens to $f(x)$. If there is a number L such that $f(x)$ gets arbitrarily close to L if one chooses x sufficiently large, then we write

$$\lim_{x\to\infty} f(x) = L, \quad \text{or} \quad \lim_{x\uparrow\infty} f(x) = L, \quad \text{or} \quad \lim_{x\nearrow\infty} f(x) = L.$$

("The limit for x going to infinity is L.")

4.5. Example – Limit of $1/x$. The larger you choose x, the smaller its reciprocal $1/x$ becomes. Therefore, it seems reasonable to say

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

Here is the precise definition:

4.6. Definition of limit at ∞. *Let f be some function which is defined on some interval $x_0 < x < \infty$. If there is a number L such that for every $\varepsilon > 0$ one can find an A such that*

$$x > A \implies |f(x) - L| < \varepsilon$$

for all x, then we say that the limit of $f(x)$ for $x \to \infty$ is L.

The definition is very similar to the original definition of the limit. Instead of δ which specifies how close x should be to a, we now have a number A which says how large x should be, which is a way of saying "how close x should be to infinity."

4.7. Example – Limit of $1/x$ (again) . To ***prove*** that $\lim_{x\to\infty} 1/x = 0$ we apply the definition to $f(x) = 1/x$, $L = 0$.

For given $\varepsilon > 0$ we need to show that

$$\left|\frac{1}{x} - L\right| < \varepsilon \text{ for all } x > A \tag{10}$$

provided we choose the right A.

How do we choose A? A is not allowed to depend on x, but it may depend on ε.

If we assume for now that we will only consider positive values of x, then (10) simplifies to

$$\frac{1}{x} < \varepsilon$$

which is equivalent to

$$x > \frac{1}{\varepsilon}.$$

This tells us how to choose A. Given any positive ε, we will simply choose

$$A = \frac{1}{\varepsilon}$$

Then one has $|\frac{1}{x} - 0| = \frac{1}{x} < \varepsilon$ for all $x > A$. Hence we have proved that $\lim_{x\to\infty} 1/x = 0$.

5. Properties of the Limit

The precise definition of the limit is not easy to use, and fortunately we won't use it very often in this class. Instead, there are a number of properties that limits have which allow you to compute them without having to resort to "epsiloncy."

The following properties also apply to the variations on the limit from 4. I.e. the following statements remain true if one replaces each limit by a one-sided limit, or a limit for $x \to \infty$.

Limits of constants and of x. If a and c are constants, then

$$\lim_{x\to a} c = c \tag{P_1}$$

and

$$\lim_{x\to a} x = a. \tag{P_2}$$

Limits of sums, products and quotients. Let F_1 and F_2 be two given functions whose limits for $x \to a$ we know,

$$\lim_{x\to a} F_1(x) = L_1, \qquad \lim_{x\to a} F_2(x) = L_2.$$

Then

$$\lim_{x\to a} \big(F_1(x) + F_2(x)\big) = L_1 + L_2, \tag{P_3}$$

$$\lim_{x\to a} \big(F_1(x) - F_2(x)\big) = L_1 - L_2, \tag{P_4}$$

$$\lim_{x\to a} \big(F_1(x) \cdot F_2(x)\big) = L_1 \cdot L_2 \tag{P_5}$$

Finally, if $\lim_{x\to a} F_2(x) \neq 0$,

$$\lim_{x\to a} \frac{F_1(x)}{F_2(x)} = \frac{L_1}{L_2}. \tag{P_6}$$

In other words the limit of the sum is the sum of the limits, etc. One can prove these laws using the definition of limit in §2 but we will not do this here. However, I hope these laws seem like common sense: if, for x close to a, the quantity $F_1(x)$ is close to L_1 and $F_2(x)$ is close to L_2, then certainly $F_1(x) + F_2(x)$ should be close to $L_1 + L_2$.

There are two more properties of limits which we will add to this list later on. They are the "Sandwich Theorem" (§9) and the substitution theorem (§10).

6. Examples of limit computations

6.1. Find $\lim_{x\to 2} x^2$. One has

$$\begin{aligned}\lim_{x\to 2} x^2 &= \lim_{x\to 2} x\cdot x \\ &= \big(\lim_{x\to 2} x\big)\cdot\big(\lim_{x\to 2} x\big) && \text{by } (P_5) \\ &= 2\cdot 2 = 4.\end{aligned}$$

Similarly,

$$\begin{aligned}\lim_{x\to 2} x^3 &= \lim_{x\to 2} x\cdot x^2 \\ &= \big(\lim_{x\to 2} x\big)\cdot\big(\lim_{x\to 2} x^2\big) && (P_5)\text{ again} \\ &= 2\cdot 4 = 8,\end{aligned}$$

and, by (P_4)

$$\lim_{x\to 2} x^2 - 1 = \lim_{x\to 2} x^2 - \lim_{x\to 2} 1 = 4 - 1 = 3,$$

and, by (P_4) again,

$$\lim_{x\to 2} x^3 - 1 = \lim_{x\to 2} x^3 - \lim_{x\to 2} 1 = 8 - 1 = 7,$$

Putting all this together, one gets

$$\lim_{x\to 2}\frac{x^3-1}{x^2-1} = \frac{2^3-1}{2^2-1} = \frac{8-1}{4-1} = \frac{7}{3}$$

because of (P_6). To apply (P_6) we must check that the denominator ("L_2") is not zero. Since the denominator is 3 everything is OK, and we were allowed to use (P_6).

6.2. Try the examples 1.3 and 1.4 using the limit properties. To compute $\lim_{x\to 2}(x^2-2x)/(x^2-4)$ we first use the limit properties to find

$$\lim_{x\to 2} x^2 - 2x = 0 \text{ and } \lim_{x\to 2} x^2 - 4 = 0.$$

to complete the computation we would like to apply the last property (P_6) about quotients, but this would give us

$$\lim_{x\to 2} f(x) = \frac{0}{0}.$$

The denominator is zero, so we were not allowed to use (P_6) (and the result doesn't mean anything anyway). We have to do something else.

The function we are dealing with is a ***rational function***, which means that it is the quotient of two polynomials. For such functions there is an algebra trick which always allows you to compute the limit even if you first get $\frac{0}{0}$. The thing to do is to divide numerator and denominator by $x-2$. In our case we have

$$x^2 - 2x = (x-2)\cdot x, \qquad x^2 - 4 = (x-2)\cdot(x+2)$$

so that

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\frac{(x-2)\cdot x}{(x-2)\cdot(x+2)} = \lim_{x\to 2}\frac{x}{x+2}.$$

After this simplification we ***can*** use the properties $(P_{\ldots})$ to compute

$$\lim_{x\to 2} f(x) = \frac{2}{2+2} = \frac{1}{2}.$$

6.3. Example – Find $\lim_{x\to 2}\sqrt{x}$. Of course, you would think that $\lim_{x\to 2}\sqrt{x} = \sqrt{2}$ and you can indeed prove this using δ & ε (See problem 44.) But is there an easier way? There is nothing in the limit properties which tells us how to deal with a square root, and using them we can't even prove that there is a limit. However, if you ***assume*** that the limit exists then the limit properties allow us to find this limit.

The argument goes like this: suppose that there is a number L with

$$\lim_{x\to 2}\sqrt{x} = L.$$

Then property (P_5) implies that

$$L^2 = \bigl(\lim_{x\to 2}\sqrt{x}\bigr)\cdot\bigl(\lim_{x\to 2}\sqrt{x}\bigr) = \lim_{x\to 2}\sqrt{x}\cdot\sqrt{x} = \lim_{x\to 2} x = 2.$$

In other words, $L^2 = 2$, and hence L must be either $\sqrt{2}$ or $-\sqrt{2}$. We can reject the latter because whatever x does, its squareroot is always a positive number, and hence it can never "get close to" a negative number like $-\sqrt{2}$.

Our conclusion: if the limit exists, then

$$\lim_{x\to 2}\sqrt{x} = \sqrt{2}.$$

The result is not surprising: if x gets close to 2 then $\sqrt{x}$ gets close to $\sqrt{2}$.

6.4. Example – The derivative of $\sqrt{x}$ at $x = 2$. Find

$$\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2}$$

assuming the result from the previous example.

Solution: The function is a fraction whose numerator and denominator vanish when $x = 2$, i.e. the limit is of the form $\frac{0}{0}$. We use the same algebra trick as before, namely we factor numerator and denominator:

$$\frac{\sqrt{x}-\sqrt{2}}{x-2} = \frac{\sqrt{x}-\sqrt{2}}{(\sqrt{x}-\sqrt{2})(\sqrt{x}+\sqrt{2})} = \frac{1}{\sqrt{x}+\sqrt{2}}.$$

Now one can use the limit properties to compute

$$\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2} = \lim_{x\to 2}\frac{1}{\sqrt{x}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}.$$

6.5. Limit as $x\to\infty$ of rational functions. A rational function is the quotient of two polynomials, so

$$R(x) = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}. \tag{11}$$

We have seen that

$$\lim_{x\to\infty}\frac{1}{x} = 0$$

We even proved this in example 4.7. Using this you can find the limit at ∞ for any rational function $R(x)$ as in (11). One could turn the outcome of the calculation of $\lim_{x\to\infty} R(x)$ into a recipe/formula involving the degrees n and m of the numerator and denominator, and also their coefficients a_i, b_j, which students would then memorize, but it is better to remember "the trick."

To find $\lim_{x\to\infty} R(x)$ divide numerator and denominator by x^m (the highest power of x occurring in the denominator).

For example, let's compute

$$\lim_{x\to\infty}\frac{3x^2+3}{5x^2+7x-39}.$$

Remember the trick and divide top and bottom by x^2, and you get

$$\begin{aligned}\lim_{x\to\infty} \frac{3x^2+3}{5x^2+7x-39} &= \lim_{x\to\infty} \frac{3+3/x^2}{5+7/x-39/x^2} \\ &= \frac{\lim_{x\to\infty} 3+3/x^2}{\lim_{x\to\infty} 5+7/x-39/x^2} \\ &= \frac{3}{5}\end{aligned}$$

Here we have used the limit properties (P_*) to break the limit down into little pieces like $\lim_{x\to\infty} 39/x^2$ which we can compute as follows

$$\lim_{x\to\infty} 39/x^2 = \lim_{x\to\infty} 39\cdot\left(\frac{1}{x}\right)^2 = \left(\lim_{x\to\infty} 39\right)\cdot\left(\lim_{x\to\infty}\frac{1}{x}\right)^2 = 39\cdot 0^2 = 0.$$

6.6. Another example with a rational function . Compute

$$\lim_{x\to\infty} \frac{x}{x^3+5}.$$

We apply "the trick" again and divide numerator and denominator by x^3. This leads to

$$\lim_{x\to\infty} \frac{x}{x^3+5} = \lim_{x\to\infty} \frac{1/x^2}{1+5/x^3} = \frac{\lim_{x\to\infty} 1/x^2}{\lim_{x\to\infty} 1+5/x^3} = \frac{0}{1} = 0.$$

To show all possible ways a limit of a rational function can turn out we should do yet another example, but that one belongs in the next section (see example 7.6.)

7. When limits fail to exist

In the last couple of examples we worried about the possibility that a limit $\lim_{x\to a} g(x)$ actually might not exist. This can actually happen, and in this section we'll see a few examples of what failed limits look like. First let's agree on what we will call a "failed limit."

7.1. Definition. *If there is no number L such that $\lim_{x\to a} f(x) = L$, then we say that the limit $\lim_{x\to a} f(x)$ does not exist.*

7.2. The sign function near $x = 0$. The "sign function[1]" is defined by

$$\operatorname{sign}(x) = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$$

Note that "the sign of zero" is defined to be zero. But does the sign function have a limit at $x = 0$, i.e. does $\lim_{x\to 0} \operatorname{sign}(x)$ exist? And is it also zero? The answers are ***no*** and ***no***, and here is why: suppose that for some number L one had

$$\lim_{x\to 0} \operatorname{sign}(x) = L,$$

then since for arbitrary small positive values of x one has $\operatorname{sign}(x) = +1$ one would think that $L = +1$. But for arbitrarily small negative values of x one has $\operatorname{sign}(x) = -1$, so one would conclude that $L = -1$. But one number L can't be both $+1$ and -1 at the same time, so there is no such L, i.e. there is no limit.

$$\lim_{x\to 0} \operatorname{sign}(x) \text{ does not exist.}$$

[1]Some people don't like the notation $\operatorname{sign}(x)$, and prefer to write

$$g(x) = \frac{x}{|x|}$$

instead of $g(x) = \operatorname{sign}(x)$. If you think about this formula for a moment you'll see that $\operatorname{sign}(x) = x/|x|$ for all $x \neq 0$. When $x = 0$ the quotient $x/|x|$ is of course not defined.

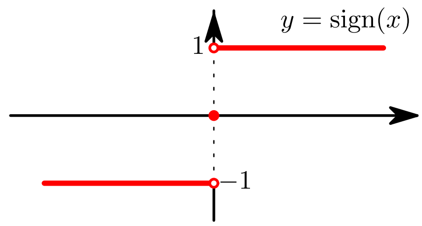


Figure 1. The sign function.

In this example the one-sided limits do exist, namely,

$$\lim_{x \searrow 0} \text{sign}(x) = 1 \text{ and } \lim_{x \nearrow 0} \text{sign}(x) = -1.$$

All this says is that when x approaches 0 through positive values, its sign approaches $+1$, while if x goes to 0 through negative values, then its sign approaches -1.

7.3. The example of the backward sine. Contemplate the limit as $x \to 0$ of the "backward sine," i.e.

$$\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right).$$

When $x = 0$ the function $f(x) = \sin(\pi/x)$ is not defined, because its definition involves division by x. What happens to $f(x)$ as $x \to 0$? First, π/x becomes larger and larger ("goes to infinity") as $x \to 0$. Then, taking the sine, we see that $\sin(\pi/x)$ oscillates between $+1$ and -1 infinitely often as $x \to 0$. This means that $f(x)$ gets close to any number between -1 and $+1$ as $x \to 0$, but that the function $f(x)$ ***never stays close*** to any particular value because it keeps oscillating up and down.

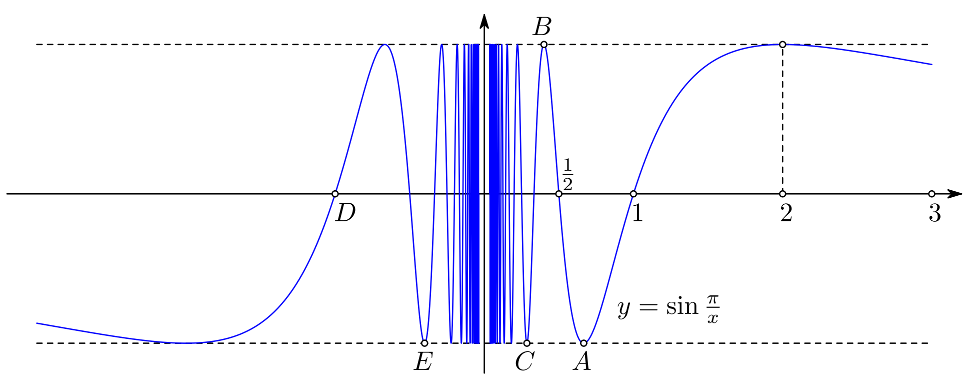


Figure 2. Graph of $y = \sin \frac{\pi}{x}$ for $-3 < x < 3$, $x \neq 0$.

Here again, the limit $\lim_{x \to 0} f(x)$ does not exist. We have arrived at this conclusion by only considering what $f(x)$ does for small positive values of x. So the limit fails to exist in a stronger way than in the example of the sign-function. There, even though the limit didn't exist, the one-sided limits existed. In the present example we see that even the one-sided limit

$$\lim_{x \searrow 0} \sin \frac{\pi}{x}$$

does not exist.

7.4. Trying to divide by zero using a limit. The expression $1/0$ is not defined, but what about

$$\lim_{x\to 0} \frac{1}{x}?$$

This limit also does not exist. Here are two reasons:

It is common wisdom that if you divide by a small number you get a large number, so as $x \searrow 0$ the quotient $1/x$ will not be able to stay close to any particular finite number, and the limit can't exist.

"Common wisdom" is not always a reliable tool in mathematical proofs, so here is a better argument. The limit can't exist, because that would contradict the limit properties $(P_1)\cdots(P_6)$. Namely, suppose that there were an number L such that

$$\lim_{x\to 0} \frac{1}{x} = L.$$

Then the limit property (P_5) would imply that

$$\lim_{x\to 0}\bigl(\frac{1}{x}\cdot x\bigr) = \bigl(\lim_{x\to 0}\frac{1}{x}\bigr)\cdot\bigl(\lim_{x\to 0} x\bigr) = L\cdot 0 = 0.$$

On the other hand $\frac{1}{x}\cdot x = 1$ so the above limit should be 1! A number can't be both 0 and 1 at the same time, so we have a contradiction. The assumption that $\lim_{x\to 0} 1/x$ exists is to blame, so it must go.

7.5. Using limit properties to show a limit does *not* exist. The limit properties tell us how to prove that certain limits exist (and how to compute them). Although it is perhaps not so obvious at first sight, they also allow you to prove that certain limits do not exist. The previous example shows one instance of such use. Here is another.

Property (P_3) says that if both $\lim_{x\to a} g(x)$ and $\lim_{x\to a} h(x)$ exist then $\lim_{x\to a} g(x) + h(x)$ also must exist. You can turn this around and say that if $\lim_{x\to a} g(x) + h(x)$ does not exist then either $\lim_{x\to a} g(x)$ or $\lim_{x\to a} h(x)$ does not exist (or both limits fail to exist).

For instance, the limit

$$\lim_{x\to 0} \frac{1}{x} - x$$

can't exist, for if it did, then the limit

$$\lim_{x\to 0}\frac{1}{x} = \lim_{x\to 0}\bigl(\frac{1}{x} - x + x\bigr) = \lim_{x\to 0}\bigl(\frac{1}{x} - x\bigr) + \lim_{x\to 0} x$$

would also have to exist, and we know $\lim_{x\to 0}\frac{1}{x}$ doesn't exist.

7.6. Limits at ∞ which don't exist. If you let x go to ∞, then x will not get "closer and closer" to any particular number L, so it seems reasonable to guess that

$$\lim_{x\to\infty} x \text{ does not exist.}$$

One can prove this from the limit definition (and see exercise 72).

Let's consider

$$L = \lim_{x\to\infty}\frac{x^2+2x-1}{x+2}.$$

Once again we divide numerator and denominator by the highest power in the denominator (i.e. x)

$$L = \lim_{x\to\infty}\frac{x+2-\frac{1}{x}}{1+2/x}$$

Here the denominator has a limit ('tis 1), but the numerator does not, for if $\lim_{x\to\infty} x+2-\frac{1}{x}$ existed then, since $\lim_{x\to\infty}(2-1/x) = 2$ exists,

$$\lim_{x\to\infty} x = \lim_{x\to\infty}\Bigl[\bigl(x+2-\frac{1}{x}\bigr) - \bigl(2-\frac{1}{x}\bigr)\Bigr]$$

would also have to exist, and $\lim_{x\to\infty} x$ doesn't exist.

So we see that L is the limit of a fraction in which the denominator has a limit, but the numerator does not. In this situation the limit L itself can never exist. If it did, then

$$\lim_{x\to\infty}\left(x+2-\frac{1}{x}\right) = \lim_{x\to\infty}\frac{x+2-\frac{1}{x}}{1+2/x}\cdot(1+2/x)$$

would also have to have a limit.

8. What's in a name?

There is a big difference between the variables x and a in the formula

$$\lim_{x\to a} 2x+1,$$

namely a is a ***free variable***, while x is a ***dummy variable*** (or "placeholder" or a "bound variable.")

The difference between these two kinds of variables is this:

- if you replace a dummy variable in some formula consistently by some other variable then the value of the formula does not change. On the other hand, it never makes sense to substitute a number for a dummy variable.
- the value of the formula may depend on the value of the free variable.

To understand what this means consider the example $\lim_{x\to a} 2x+1$ again. The limit is easy to compute:

$$\lim_{x\to a} 2x+1 = 2a+1.$$

If we replace x by, say u (systematically) then we get

$$\lim_{u\to a} 2u+1$$

which is again equal to $2a+1$. This computation says that *if some number gets close to a then two times that number plus one gets close to* $2a+1$. This is a very wordy way of expressing the formula, and you can shorten things by giving a name (like x or u) to the number which approaches a. But the result of our computation shouldn't depend on the name we choose, i.e. it doesn't matter if we call it x or u.

Since the name of the variable x doesn't matter it is called a dummy variable. Some prefer to call x a bound variable, meaning that in

$$\lim_{x\to a} 2x+1$$

the x in the expression $2x+1$ is bound to the x written underneath the limit – you can't change one without changing the other.

Substituting a number for a dummy variable usually leads to complete nonsense. For instance, let's try setting $x=3$ in our limit, i.e. what is

$$\lim_{3\to a} 2\cdot 3+1\,?$$

Of course $2\cdot 3+1=7$, but what does 7 do when 3 gets closer and closer to the number a? That's a silly question, because 3 is a constant and it doesn't "get closer" to some other number like a! If you ever see 3 get closer to another number then it's time to take a vacation.

On the other hand the variable a is free: you can assign it particular values, and its value will affect the value of the limit. For instance, if we set $a=3$ (but leave x alone) then we get

$$\lim_{x\to 3} 2x+1$$

and there's nothing strange about that (the limit is $2\cdot 3+1=7$, no problem.) You could substitute other values of a and you would get a different answer. In general you get $2a+1$.

9. Limits and Inequalities

This section has two theorems which let you compare limits of different functions. The properties in these theorems are not formulas that allow you to compute limits like the properties $(P_1)\dots(P_6)$ from §5. Instead, they allow you to ***reason*** about limits, i.e. they let you say that this or that limit is positive, or that it must be the same as some other limit which you find easier to think about.

The first theorem should not surprise you – all it says is that bigger functions have bigger limits.

9.1. Theorem. *Let f and g be functions whose limits for $x \to a$ exist, and assume that $f(x) \le g(x)$ holds for all x. Then*

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

A useful special case arises when you set $f(x) = 0$. The theorem then says that if a function g never has negative values, then its limit will also never be negative.

The statement may seem obvious, but it still needs a proof, starting from the ε-δ definition of limit. This will be done in lecture.

Here is the second theorem about limits and inequalities.

9.2. The Sandwich Theorem. *Suppose that*

$$f(x) \le g(x) \le h(x)$$

(for all x) and that

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x).$$

Then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \lim_{x\to a} h(x).$$

The theorem is useful when you want to know the limit of g, and when you can ***sandwich*** it between two functions f and h whose limits are easier to compute. The Sandwich Theorem looks like the first theorem of this section, but there is an important difference: in the Sandwich Theorem you don't have to assume that the limit of g exists. The inequalities $f \le g \le h$ combined with the circumstance that f and h have the same limit are enough to guarantee that the limit of g exists.

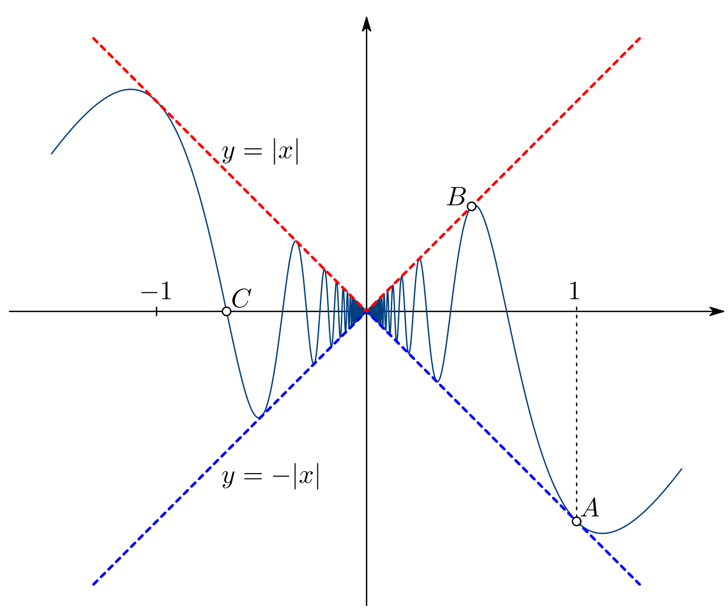


Figure 3. Graphs of $|x|$, $-|x|$ and $x\cos\frac{\pi}{x}$ for $-1.2 < x < 1.2$

9.3. Example: a Backward Cosine Sandwich. The Sandwich Theorem says that if the function $g(x)$ is sandwiched between two functions $f(x)$ and $h(x)$ and the limits of the outside functions f and h exist and are equal, then the limit of the inside function g exists and equals this common value. For example

$$-|x| \le x\cos\frac{1}{x} \le |x|$$

since the cosine is always between -1 and 1. Since

$$\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0$$

the sandwich theorem tells us that

$$\lim_{x\to 0} x\cos\frac{1}{x} = 0.$$

Note that the limit $\lim_{x\to 0}\cos(1/x)$ does ***not*** exist, for the same reason that the "backward sine" did not have a limit for $x \to 0$ (see example 7.3). Multiplying with x changed that.

10. Continuity

10.1. Definition. *A function g is **continuous** at a if*

$$\lim_{x\to a} g(x) = g(a) \tag{12}$$

A function is continuous if it is continuous at every a in its domain.

Note that when we say that a function is continuous on some interval it is understood that the domain of the function includes that interval. For example, the function $f(x) = 1/x^2$ is continuous on the interval $1 < x < 5$ but is ***not*** continuous on the interval $-1 < x < 1$.

10.2. Polynomials are continuous. For instance, let us show that $P(x) = x^2 + 3x$ is continuous at $x = 2$. To show that you have to prove that

$$\lim_{x\to 2} P(x) = P(2),$$

i.e.

$$\lim_{x\to 2} x^2 + 3x = 2^2 + 3\cdot 2.$$

You can do this two ways: using the definition with ε and δ (i.e. the hard way), or using the limit properties $(P_1)\dots(P_6)$ from §5 (just as good, and easier, even though it still takes a few lines to write it out – do both!)

10.3. Rational functions are continuous. Let $R(x) = \frac{P(x)}{Q(x)}$ be a rational function, and let a be any number in the domain of R, i.e. any number for which $Q(a) \neq 0$. Then one has

$$\begin{aligned}
\lim_{x\to a} R(x) &= \lim_{x\to a}\frac{P(x)}{Q(x)} \\
&= \frac{\lim_{x\to a} P(x)}{\lim_{x\to a} Q(x)} && \text{property } (P_6) \\
&= \frac{P(a)}{Q(a)} && P \text{ and } Q \text{ are continuous} \\
&= R(a).
\end{aligned}$$

This shows that R is indeed continuous at a.

10.4. Some discontinuous functions. If $\lim_{x\to a} g(x)$ does not exist, then it certainly cannot be equal to $g(a)$, and therefore any failed limit provides an example of a discontinuous function.

For instance, the sign function $g(x) = \operatorname{sign}(x)$ from example **??** is not continuous at $x = 0$.

Is the backward sine function $g(x) = \sin(1/x)$ from example **??** also discontinuous at $x = 0$? No, it is not, for two reasons: first, the limit $\lim_{x\to 0}\sin(1/x)$ does not exist, and second, we haven't even defined the function $g(x)$ at $x = 0$, so even if the limit existed, we would have no value $g(0)$ to compare it with.

10.5. How to make functions discontinuous. Here is a discontinuous function:

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 3, \\ 47 & \text{if } x = 3. \end{cases}$$

In other words, we take a continuous function like $g(x) = x^2$, and change its value somewhere, e.g. at $x = 3$. Then

$$\lim_{x\to 3} f(x) = 9 \neq 47 = f(3).$$

The reason that the limit is 9 is that our new function $f(x)$ coincides with our old continuous function $g(x)$ for all x except $x = 3$. Therefore the limit of $f(x)$ as $x \to 3$ is the same as the limit of $g(x)$ as $x \to 3$, and since g is continuous this is $g(3) = 9$.

10.6. Sandwich in a bow tie. We return to the function from example **??**. Consider

$$f(x) = \begin{cases} x\cos\left(\frac{1}{x}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$

Then f is continuous at $x = 0$ by the Sandwich Theorem (see Example **??**).

If we change the definition of f by picking a different value at $x = 0$ the new function will not be continuous, since changing f at $x = 0$ does not change the limit $\lim_{x\to 0} f(x)$. Since this limit is zero, $f(0) = 0$ is the only possible choice of $f(0)$ which makes f continuous at $x = 0$.

11. Substitution in Limits

Given two functions f and g one can consider their composition $h(x) = f(g(x))$. To compute the limit

$$\lim_{x\to a} f\big(g(x)\big)$$

we write $u = g(x)$, so that we want to know

$$\lim_{x\to a} f(u) \text{ where } u = g(x).$$

Suppose that you can find the limits

$$L = \lim_{x\to a} g(x) \text{ and } \lim_{u\to L} f(u) = M.$$

Then it seems reasonable that as x approaches a, $u = g(x)$ will approach L, and $f(g(x))$ approaches M.

This is in fact a theorem:

11.1. Theorem. *If $\lim_{x\to a} g(x) = a$, and if the function f is continuous at $u = L$, then*

$$\lim_{x\to a} f\big(g(x)\big) = \lim_{u\to L} f(u) = f(L).$$

Another way to write this is

$$\lim_{x\to a} f\big(g(x)\big) = f\big(\lim g(x)\big).$$

11.2. Example: compute $\lim_{x\to 3} \sqrt{x^3 - 3x^2 + 2}$. The given function is the composition of two functions, namely

$$\sqrt{x^3 - 3x^2 + 2} = \sqrt{u}, \text{ with } u = x^3 - 3x^2 + 2,$$

or, in function notation, we want to find $\lim_{x\to 3} h(x)$ where

$$h(x) = f(g(x)), \text{ with } g(x) = x^3 - 3x^2 + 2 \text{ and } g(x) = \sqrt{x}.$$

Either way, we have

$$\lim_{x\to 3} x^3 - 3x^2 + 2 = 2 \quad \text{and} \quad \lim_{u\to 2} \sqrt{u} = \sqrt{2}.$$

You get the first limit from the limit properties $(P_1)\dots(P_5)$. The second limit says that taking the square root is a continuous function, which it is. We have not proved that (yet), but this particular limit is the one from example 6.3. Putting these two limits together we conclude that the limit is $\sqrt{2}$.

Normally, you write this whole argument as follows:

$$\lim_{x\to 3} \sqrt{x^3 - 3x^2 + 2} = \sqrt{\lim_{x\to 3} x^3 - 3x^2 + 2} = \sqrt{2},$$

where you must point out that $f(x) = \sqrt{x}$ is a continuous function to justify the first step.

Another possible way of writing this is

$$\lim_{x\to 3} \sqrt{x^3 - 3x^2 + 2} = \lim_{u\to 2} \sqrt{u} = \sqrt{2},$$

where you must say that you have substituted $u = x^3 - 3x^2 + 2$.

12. Exercises

Find the following limits.

52. $\lim_{x\to -7} (2x + 5)$

53. $\lim_{x\to 7-} (2x + 5)$

54. $\lim_{x\to -\infty} (2x + 5)$

55. $\lim_{x\to -4} (x + 3)^{2006}$

56. $\lim_{x\to -4} (x + 3)^{2007}$

57. $\lim_{x\to -\infty} (x + 3)^{2007}$

58. $\lim_{t\to 1} \dfrac{t^2 + t - 2}{t^2 - 1}$

59. $\lim_{t\nearrow 1} \dfrac{t^2 + t - 2}{t^2 - 1}$

60. $\lim_{t\to -1} \dfrac{t^2 + t - 2}{t^2 - 1}$

61. $\lim_{x\to \infty} \dfrac{x^2 + 3}{x^2 + 4}$

62. $\lim_{x\to \infty} \dfrac{x^5 + 3}{x^2 + 4}$

63. $\lim_{x\to \infty} \dfrac{x^2 + 1}{x^5 + 2}$

64. $\lim_{x\to \infty} \dfrac{(2x + 1)^4}{(3x^2 + 1)^2}$

65. $\lim_{u\to \infty} \dfrac{(2u + 1)^4}{(3u^2 + 1)^2}$

66. $\lim_{t\to 0} \dfrac{(2t + 1)^4}{(3t^2 + 1)^2}$

67. What are the coordinates of the points labeled A, ..., E in Figure 2 (the graph of $y = \sin \pi/x$).

68. If $\lim_{x\to a} f(x)$ exists then f is continuous at $x = a$. *True or false?*

69. Give two examples of functions for which $\lim_{x\searrow 0} f(x)$ does not exist.

70. Group Problem.

If $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ both do not exist, then $\lim_{x\to 0}\big(f(x) + g(x)\big)$ also does not exist. *True or false?*

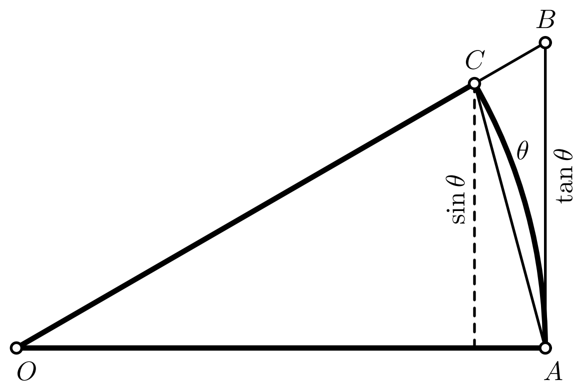
71. Group Problem.

If $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ both do not exist, then $\lim_{x\to 0}(f(x)/g(x))$ also does not exist. *True or false?*

72. Group Problem.

In the text we proved that $\lim_{x\to\infty} \frac{1}{x} = 0$. Show that this implies that $\lim_{x\to\infty} x$ does not exist. Hint: Suppose $\lim_{x\to\infty} x = L$ for some number L. Apply the limit properties to $\lim_{x\to\infty} x \cdot \dfrac{1}{x}$.

73. Evaluate $\lim_{x\to 9} \dfrac{\sqrt{x} - 3}{x - 9}$. Hint: Multiply top and bottom by $\sqrt{x} + 3$.



Proving $\lim_{x\to 0} \frac{\sin\theta}{\theta} = 1$

The circular wedge OAC contains the triangle OAC and is contained in the right triangle OAB.

The area of triangle OAC is $\frac{1}{2}\sin\theta$.

The area of circular wedge OAC is $\frac{1}{2}\theta$.

The area of right triangle OAB is $\frac{1}{2}\tan\theta$.

Hence one has $\sin\theta < \theta < \tan\theta$ for all angles $0 < \theta < \pi/2$.

74. Evaluate $\lim_{x\to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x-2}$.

75. Evaluate $\lim_{x\to 2} \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{2}}}{x-2}$.

76. A function f is defined by

$$f(x) = \begin{cases} x^3 & \text{for } x < -1 \\ ax+b & \text{for } -1 \le x < 1 \\ x^2+2 & \text{for } x \ge 1. \end{cases}$$

where a and b are constants. The function f is continuous. What are a and b?

77. Find a constant k such that the function

$$f(x) = \begin{cases} 3x+2 & \text{for } x < 2 \\ x^2+k & \text{for } x \ge 2. \end{cases}$$

is continuous. Hint: Compute the one-sided limits.

78. Find constants a and c such that the function

$$f(x) = \begin{cases} x^3+c & \text{for } x < 0 \\ ax+c^2 & \text{for } 0 \le x < 1 \\ \arctan x & \text{for } x \ge 1. \end{cases}$$

is continuous for all x.

13. Two Limits in Trigonometry

In this section we'll derive a few limits involving the trigonometric functions. You can think of them as saying that for small angles θ one has

$$\sin\theta \approx \theta \quad \text{and} \quad \cos\theta \approx 1 - \tfrac{1}{2}\theta^2.$$

We will use these limits when we compute the derivatives of Sine, Cosine and Tangent.

13.1. Theorem. $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$

Proof. The proof requires a few sandwiches and some geometry.

We begin by only considering positive angles, and in fact we will only consider angles $0 < \theta < \pi/2$.

Since the wedge OAC contains the triangle OAC its area must be larger. The area of the wedge is $\frac{1}{2}\theta$ and the area of the triangle is $\frac{1}{2}\sin\theta$, so we find that

$$0 < \sin\theta < \theta \text{ for } 0 < \theta < \frac{\pi}{2}. \tag{13}$$

The Sandwich Theorem implies that

$$\lim_{\theta\searrow 0} \sin\theta = 0. \tag{14}$$

Moreover, we also have

$$\lim_{\theta\searrow 0} \cos\theta = \lim_{\theta\searrow 0} \sqrt{1-\sin^2\theta} = 1. \tag{15}$$

Next we compare the areas of the wedge OAC and the larger triangle OAB. Since OAB has area $\frac{1}{2}\tan\theta$ we find that

$$\theta < \tan\theta$$

for $0 < \theta < \frac{\pi}{2}$. Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$ we can multiply with $\cos\theta$ and divide by θ to get

$$\cos\theta < \frac{\sin\theta}{\theta} \text{ for } 0 < \theta < \frac{\pi}{2}$$

If we go back to (15) and divide by θ, then we get

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

The Sandwich Theorem can be used once again, and now it gives

$$\lim_{\theta\searrow 0} \frac{\sin\theta}{\theta} = 1.$$

This is a one-sided limit. To get the limit in which $\theta \nearrow 0$, you use that $\sin\theta$ is an odd function. □

13.2. An example. We will show that

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta^2} = \frac{1}{2}. \tag{16}$$

This follows from $\sin^2\theta + \cos^2\theta = 1$. Namely,

$$\begin{aligned}\frac{1-\cos\theta}{\theta^2} &= \frac{1}{1+\cos\theta}\,\frac{1-\cos^2\theta}{\theta^2}\\ &= \frac{1}{1+\cos\theta}\,\frac{\sin^2\theta}{\theta^2}\\ &= \frac{1}{1+\cos\theta}\left\{\frac{\sin\theta}{\theta}\right\}^2.\end{aligned}$$

We have just shown that $\cos\theta \to 1$ and $\frac{\sin\theta}{\theta} \to 1$ as $\theta \to 0$, so (16) follows.

14. Exercises

Find each of the following limits or show that it does not exist. Distinguish between limits which are infinite and limits which do not exist.

79. $\lim_{\alpha\to 0} \frac{\sin 2\alpha}{\sin\alpha}$ (two ways: with and without the double angle formula!)

80. $\lim_{x\to 0} \frac{\sin 3x}{2x}$.

81. $\lim_{\theta\to 0} \frac{\tan\theta}{\theta}$.

82. $\lim_{\alpha\to 0} \frac{\tan 4\alpha}{\sin 2\alpha}$.

83. $\lim_{x\to 0} \frac{1-\cos x}{x\sin x}$.

84. $\lim_{\theta\to\pi/2} \frac{1-\sin\theta}{\theta-\pi/2}$

85. $\lim_{x\to\infty} \frac{2x^3+3x^2\cos x}{(x+2)^3}$.

86. $\lim_{x\to 0} \frac{\sin^2 x}{1-\cos x}$.

87. $\lim_{x\to 0} \frac{\sin(x^2)}{x^2}$.

88. $\lim_{x\to 0} \frac{x(1-\cos x)}{\tan^3 x}$.

89. $\lim_{x\to 0} \frac{\sin(x^2)}{1-\cos x}$.

90. $\lim_{x\to\pi/2} \frac{x-\frac{\pi}{2}}{\cos x}$.

91. $\lim_{x\to\pi/2} (x-\frac{\pi}{2})\tan x$.

92. $\lim_{x\to 0} \frac{\cos x}{x^2+9}$.

93. $\lim_{x\to\pi} \frac{\sin x}{x-\pi}$.

94. $\lim_{x\to 0} \frac{\sin x}{x+\sin x}$.

95. $A = \lim_{x\to\infty} \frac{\sin x}{x}$. **(!!)**

97. Is there a constant k such that the function

$$f(x) = \begin{cases} \sin(1/x) & \text{for } x \neq 0 \\ k & \text{for } x = 0. \end{cases}$$

is continuous? If so, find it; if not, say why.

98. Find a constant A so that the function

$$f(x) = \begin{cases} \dfrac{\sin x}{2x} & \text{for } x \neq 0 \\ A & \text{when } x = 0 \end{cases}$$

99. Compute $\lim_{x\to\infty} x \sin \frac{\pi}{x}$ and $\lim_{x\to\infty} x \tan \frac{\pi}{x}$. (Hint: substitute something).

100. **Group Problem.**

(*Geometry & Trig review*) Let A_n be the area of the regular n-gon inscribed in the unit circle, and let B_n be the area of the regular n-gon whose inscribed circle has radius 1.

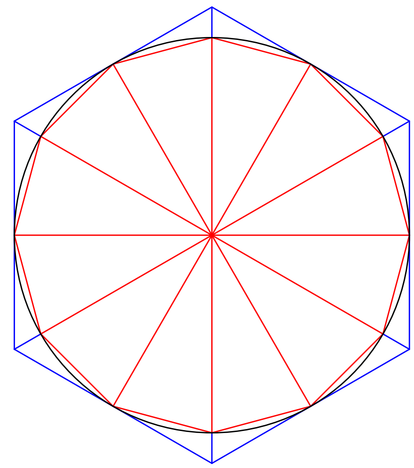
(a) Show that $A_n < \pi < B_n$.

(b) Show that

$$A_n = \frac{n}{2}\sin\frac{2\pi}{n} \text{ and } B_n = n\tan\frac{\pi}{n}$$

(c) Compute $\lim_{n\to\infty} A_n$ and $\lim_{n\to\infty} B_n$.

Here is a picture of A_{12}, B_6 and π:



On a historical note: Archimedes managed to compute A_{96} and B_{96} and by doing this got the most accurate approximation for π that was known in his time. See also:

http://www-history.mcs.st-andrews.ac.uk/HistTopics/Pi_through_the_ages.html

Since g is a differentiable function it must also be a continuous function, and hence $\lim_{x\to a} g(x) = g(a)$. So we can substitute $y = g(x)$ in the limit defining $f'(g(a))$

$$f'(g(a)) = \lim_{y\to a} \frac{f(y) - f(g(a))}{y - g(a)} = \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)}. \tag{27}$$

Put all this together and you get

$$\begin{aligned}(f \circ g)'(a) &= \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{x - a} \\ &= \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} \\ &= \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \lim_{x\to a} \frac{g(x) - g(a)}{x - a} \\ &= f'(g(a)) \ \cdot\ g'(a)\end{aligned}$$

which is what we were supposed to prove – the proof seems complete.

There is one flaw in this proof, namely, we have divided by $g(x) - g(a)$, which is not allowed when $g(x) - g(a) = 0$. This flaw can be fixed but we will not go into the details here.[2] □

13.3. First example. We go back to the functions

$$z = f(y) = y^2 + y \text{ and } y = g(x) = 2x + 1$$

from the beginning of this section. The composition of these two functions is

$$z = f(g(x)) = (2x + 1)^2 + (2x + 1) = 4x^2 + 6x + 2.$$

We can compute the derivative of this composed function, i.e. the derivative of z with respect to x in two ways. First, you simply differentiate the last formula we have:

$$\frac{dz}{dx} = \frac{d(4x^2 + 6x + 2)}{dx} = 8x + 6. \tag{28}$$

The other approach is to use the chain rule:

$$\frac{dz}{dy} = \frac{d(y^2 + y)}{dy} = 2y + 1,$$

and

$$\frac{dy}{dx} = \frac{d(2x + 1)}{dx} = 2.$$

Hence, by the chain rule one has

$$\frac{dz}{dx} = \frac{dz}{dy}\,\frac{dy}{dx} = (2y + 1) \cdot 2 = 4y + 2. \tag{29}$$

The two answers (28) and (29) should be the same. Once you remember that $y = 2x + 1$ you see that this is indeed true:

$$y = 2x + 1 \implies 4y + 2 = 4(2x + 1) + 2 = 8x + 6.$$

The two computations of dz/dx therefore lead to the same answer. In this example there was no clear advantage in using the chain rule. The chain rule becomes useful when the functions f and g become more complicated.

[2] Briefly, you have to show that the function

$$h(y) = \begin{cases} \{f(y) - f(g(a))\}/(y - g(a)) & y \neq a \\ f'(g(a)) & y = a \end{cases}$$

is continuous.

caps, then (using the same 100in^2 os sheet metal), then which choice of radius and height of the cylinder give you the container with the largest volume?

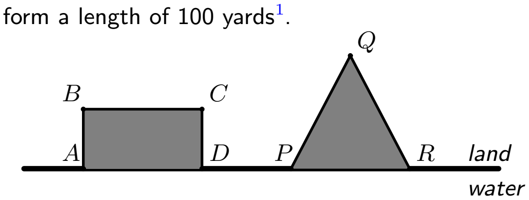
(c) Suppose you only replace the top of the cylinder with a spherical cap, and leave the bottom flat, then which choice of height and radius of the cylinder result in the largest volume?

274. A triangle has one vertex at the origin $O(0,0)$, another at the point $A(2a,0)$ and the third at $(a, a/(1+a^3))$. What are the largest and smallest areas this triangle can have if $0 \le a < \infty$?

275. Group Problem.

Queen Dido's problem

According to tradition Dido was the founder and first Queen of Carthage. When she arrived on the north coast of Africa ($\sim$800BC) the locals allowed her to take as much land as could be enclosed with the hide of one ox. She cut the hide into thin strips and put these together to form a length of 100 yards[1].



(a) If Dido wanted a rectangular region, then how wide should she choose it to enclose as much area as possible (the coastal edge of the boundary doesn't count, so in this problem the length $AB + BC + CD$ is 100 yards.)

(b) If Dido chose a region in the shape of an isosceles triangle PQR, then how wide should she make it to maximize its area (again, don't include the coast in the perimiter: $PQ + QR$ is 100 yards long, and $PQ = QR$.)

276. The product of two numbers x, y is 16. We know $x \ge 1$ and $y \ge 1$. What is the greatest possible sum of the two numbers?

277. What are the smallest and largest values that $(\sin x)(\sin y)$ can have if $x + y = \pi$ and if x and y are both nonnegative?

278. What are the smallest and largest values that $(\cos x)(\cos y)$ can have if $x + y = \frac{\pi}{2}$ and if x and y are both nonnegative?

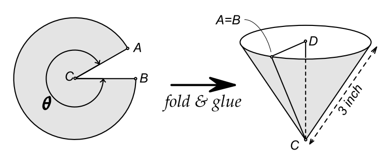
279. (a) What are the smallest and largest values that $\tan x + \tan y$ can have if $x + y = \frac{\pi}{2}$ and if x and y are both nonnegative?

(b) What are the smallest and largest values that $\tan x + 2\tan y$ can have if $x + y = \frac{\pi}{2}$ and if x and y are both nonnegative?

280. The cost per hour of fuel to run a locomotive is $v^2/25$ dollars, where v is speed (in miles per hour), and other costs are \$100 per hour regardless of speed. What is the speed that minimizes cost per mile ?

281. Group Problem.

Josh is in need of coffee. He has a circular filter with 3 inch radius. He cuts out a wedge and glues the two edges AC and BC together to make a conical filter to hold the ground coffee. The volume V of the coffee cone depends the angle θ of the piece of filter paper Josh made.



(a) Find the volume in terms of the angle θ. (Hint: how long is the circular arc AB on the left? How long is the circular top of the cone on the right? If you know that you can find the radius $AD = BD$ of the top of the cone, and also the height CD of the cone.)

(b) Which angle θ maximizes the volume V?